UDC 62-50

AN EXAMPLE OF AN IRREGULAR DIFFERENTIAL GAME

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The structure of the positional absorption and its relation to the programing constructions is investigated for a particular case by the method suggested in [1]. The paper is related to the investigations carried out in [2 — 5].

1. Let a conflict-controlled system be described by the linear equation

$$dx/dt = A(t)x + B(t)u + C(t)v \qquad (1.1)$$
$$x \in R^n, \quad u \in P, \quad v \in Q$$

where P and Q are convex compacts in R^p and R^q, respectively. We consider the problem of control over a finite time interval $[t_0, \vartheta_0]$, $t_0 < \vartheta_0$. The first player uses the control $u \in P$ and attempts to minimize the values $f_0(x[\vartheta_0])$ of the function f_0 continuous on R^n, on the trajectories of the system (1.1). The second player uses the control $v \in Q$ and persues the opposite goal. We denote by

$$c_0(t_*, x_*) = \min_{\{U\}} \max_{X(\cdot, t_*, x_*, U)} f_0(x[\vartheta_0]) = \max_{\{V\}} \min_{X(\cdot, t_*, x_*, V)} f_0(x[\vartheta_0]) \qquad (1.2)$$

the value of the game at the minimax-maximin $f_0(x[\vartheta_0])$ (see [2]). Here $\{U\}$ and $\{V\}$ are the sets of strategies of the first and second player; $X(\cdot, t_*, x_*, U)$ and $X(\cdot, t_*, x_*, V)$ are the sets of all motions from the position (t_*, x_*), generated by the strategies U and V, respectively, and defined as uniform limits of the Euler broken lines [2]. The function $c_0(t, x)$ can be found in the following manner [1]. We define on the space $C(\Lambda_n)$, $\Lambda_n = [t_0, \vartheta_0] \times R^n$ the operator Γ and the function ε° as follows:

$$\varepsilon^\circ(t_*, x_*) = \max_{\{v(\cdot), [t_*, \vartheta_0]\}} \min_{G(\vartheta_0, t_*, x_*, v(\cdot))} f_0(x)$$

$$(\Gamma(g))(t_*, x_*) = \max_{[t_*, \vartheta_0]} \max_{\{v(\cdot), [t_*, \vartheta_0]\}} \min_{G(t, t_*, x_*, v(\cdot))} g(t, x)$$

$(t_*, x_*) \in \Lambda_n$, $g \in C(\Lambda_n)$). Here $\{v(\cdot), [t_*, \vartheta_0]\}$ is the set of all measurable functions, $[t_*, \vartheta_0] \to Q$ and $G(t, t_*, x_*, v(\cdot))$ is the set of all points

$$\varphi(t, t_*, x_*, u(\cdot), (v\cdot)) = X(t, t_*)x_* + \int_{[t_*, t]} X(t, \xi)[B(\xi)u(\xi) + C(\xi)v(\xi)]\, d\xi$$

($X(t, \xi)$ is the fundamental matrix of the solutions of (1.1)), when $u(\cdot)$ traverses the set $\{u(\cdot), [t_*, \vartheta_0]\}$ of all measurable functions $[t_*, \vartheta_0] \to P$. The function c_0 is a monotonous limit of the sequence $\varepsilon^{(k)}$, $k \in N_0 = \{0, 1, \ldots\}$ defined by the condition $\varepsilon^{(k)} = \Gamma^k(\varepsilon^\circ)$, where Γ^k is the corresponding power of Γ: $c_0(t, x) = \lim_k \uparrow \varepsilon^{(k)}(t, x)$, $(t, x) \in \Lambda_n$.

Cases are known in which c_0 can be found using a finite number of iterations, i.e. $c_0 = \varepsilon^{(k)}$ for some $k \in N_0$. The aim of this paper is to analyse a specific example of the system (1.1), in which for each k there exists a position in which $c_0(t, x) = \varepsilon^{(k)}(t, x)$, as well as a position such that $c_0(t, x) \neq \varepsilon^{(k)}(t, x)$ for all k.

Let us consider the system

$$dx/dt = u + v, \quad u \in P = [-1, +1], \quad v \in Q = [-2, +2] \tag{1.3}$$
$$f_0(x) = \min_M |x - m| = d(x, M)$$
$$M = (-\infty, -1] \cup [1, \infty), \quad t_0 = 0, \ \vartheta_0 = 1$$

We denote by E_k, for every $k \in N_0$, the set of all positions $(t, x) \in \Lambda_1$, $\Lambda_1 = [0, 1] \times R^1$ for which $c_0(t, x) = \varepsilon^{(k)}(t, x)$, and by E_∞ the set of all positions $(t, x) \in \Lambda_1$ for which $c_0(t, x) \neq \varepsilon^{(k)}(t, x)$ for all $k \in N_0$. We investigate the structure of the sets E_k and E_∞, establish that the set $E_{k+1} \setminus E_k$ is nonempty for every $k \in N_0$ and that E_∞ is nonempty, and elucidate the character of the passage from E_0 to E_∞.

2. Let us determine the function ε^0 for the system (1.3). We write

$$M_1 = (-\infty, -1], \ M_2 = [1, \infty)$$
$$d(K_1, K_2) = \inf_{K_1} \inf_{K_2} |x - y|$$
$$(K_1 \subset R^1, \ K_2 \subset R^1, \ K_1 \neq \varnothing, \ K_2 \neq \varnothing)$$

Then

$$d(x, M_1) = \max(0, x + 1), \ d(x, M_2) = \max(0, 1 - x)$$
$$d(G(1, t_*, x_*, v(\cdot)), M) = \min d(G(1, t_*, x_*, v(\cdot)), M_i), \ i = 1,2$$

$$(x \in R^1, v(\cdot) \in \{v(\cdot), [t_*, 1]\}, (t_*, x_*) \in \Lambda_1)$$

It can be shown that for every $v(\cdot) \in \{v(\cdot), [t_*, 1]\}$

$$d(G(1, t_*, x_*, v(\cdot)), M) = \max\Big(0, t_* - \Big|x_* + \int_{[t_*, 1]} v(\xi)d\xi\Big|\Big) \tag{2.1}$$

Taking into account (2.1) we find, that for any position we have

$$\varepsilon^\circ(t_*, x_*) = \max\langle 0, t_* - \max[0, |x_*| - 2(1 - t_*)]\rangle \tag{2.2}$$

Let us denote by L_0 the set of all positions $(t, x) \in \Lambda_1$ for which $\varepsilon^\circ(t, x) > 0$. From (2.2) it follows that

$$L_0 = \{(t, x) : (t, x) \in (0, 1] \times R^1, \ |x| < 2 - t\} \tag{2.3}$$

To find $c_0(t_*, x_*)$, $(t_*, x_*) \in \Lambda_1$ we introduce the number $b_* = \max(0, |x_*| - (1 - t_*))$ and the set $S_* = \{(t, x) : (t, x) \in \Lambda_1, \ |x| \leqslant b_* + (1 - t)\}$. It can be shown that S_* is v-stable [2], and this implies that the strategy V_* of the second player extremal to S_* guarantees that the inequality $d(x[1], M) \geqslant \max(0, 1 - b_*)$, is true on every motion $x[\cdot] \in X(\cdot, t_*, x_*, V_*)$. The last inequality with (1.2) taken into account, yields the inequality $c_0(t_*, x_*) \geqslant \min\langle 1, \max[0, 2 - (t_* + |x_*|)]\rangle$. The converse inequality follows from the fact that in the course of forming the Euler broken lines generated by any strategy V the second player may encounter the realization $u(\cdot) \in \{u(\cdot), [t_*, 1]\}$, for which $u(t) = \operatorname{sign}(x_*)$ for all $t \in [t_*, 1]$. Therefore we have

$$c_0(t_*, x_*) = \min\langle 1, \max[0, 2 - (t_* + |x_*|)]\rangle \tag{2.4}$$

From (2.2) and (2.4) it follows that

$$E_0 = \{(t, x) : (t, x) \in \Lambda_1, \ |x| \geqslant 2(1 - t)\} \tag{2.5}$$

We can confirm by induction that for every $k \in N_0$ for all $(t,x) \in \Lambda_1 \varepsilon^{(k)}(t,x) = \varepsilon^{(k)}(t,-x)$. Moreover, the following lemma holds.

Lemma 1. For any $k \in N_0$, every position $(t, x) \in L_k$, $L_k = \{(\tau, y) : (\tau, y) \in \Lambda_1, \varepsilon^{(k)}(\tau, y) > 0\}$ and any number $b \in [0, 1]$

$$\varepsilon^{(k)}(t, bx + (1-b)(-x)) \geqslant \varepsilon^{(k)}(t, x)$$

Let $(a_k)_{k \in N_0}$ denote a sequence for which $a_0 = 2$ and

$$a_{k+1} = 2a_k/(1 + a_k) \tag{2.6}$$

for all $k \in N_0$. It can be verified that the sequence is defined correctly ($a_k + 1 \neq 0$, $k \in N_0$) and has the following properties:

1°. For every $k \in N_0$, $a_k > 1$; 2°. For every $k \in N_0$, $a_{k+1} < a_k$; 3°. The limit $\lim_k a_k = 1$ exists.

Let us set

$$S = \{(t, x) : (t, x) \in [0, 1) \times R^1, |x| \leqslant 1 - t\} \tag{2.7}$$

Lemma 2. For any $(t_*, x_*) \in S$, and $k \in N_0$ $\varepsilon^{(k)}(t_*, x_*) \neq c_0(t_*, x_*)$.

To prove this, it is sufficient to show, by virtue of (2.4), that for all $(t_*, x_*) \in S$, $k \in N_0$, $\varepsilon^{(k)}(t_*, x_*) < 1$. For $k = 0$ this follows from (2.2), and the rest is proved by induction. Lemma 2 implies that $S \subset E_\infty$.

Lemma 3. For every $k \in N_0$, the set E_k is defined by the condition

$$E_k = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_k(1 - t)\}$$

Scheme of the proof. For $k = 0$ the lemma follows from (2.5). Let

$$E_l = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_l(1 - t)\} \tag{2.8}$$

for all $l \in \{0, \ldots, m\}$, where $m \in N_0$. Then, with Lemma 2 taken into account, we have

$$E_m \subset E_{m+1} \subset S^c = \Lambda_1 \setminus S \tag{2.9}$$

Let $(t_*, x_*) \in S^c \setminus E_m$ and

$$x_0(t) = x_* - \operatorname{sign}(x_*)(t - t_*)$$

$$x^0(t) = x_* - 3\operatorname{sign}(x_*)(t - t_*)$$

$$t^\circ = \frac{a_m - (t_* + |x_*|)}{a_m - 1} \qquad \bar{t} = t_* + \frac{|x_*|}{2}$$

for every $t \in [t_*, 1]$. Then we can show that $t^0 \in (t_*, 1)$ and $\bar{t} \in [t_*, 1]$. Consider the following cases:

1°. $|x_*| \geqslant a_{m+1}(1 - t_*)$, 2°. $|x_*| < a_{m+1}(1 - t_*)$, $x_* < 0$.

1°. Using (2.8) we can confirm that $\bar{t} \geqslant t^\circ$, $(\bar{t}, x_0(t)) \in E_m$, $x^\circ(\bar{t}) = -x_0(\bar{t})$, $|x_0(\bar{t})| = |x_*|/2$ and, that for the program control $v_0(\cdot)$ of the second player satisfying the equality $v_0(t) = -2\operatorname{sign}(x_*)$ the relation $G(\bar{t}, t_*, x_*, v_0(\cdot)) = [-|x_0(\bar{t})|_*, |x_0(\bar{t})|]$ holds. Moreover, we have $c_0(t, x_0(t)) = c_0(t_*, x_*) = 2 - (t_* + |x_*|)$, $t \in [t_*, 1]$, and this implies, by virtue of Lemma 1, that

$$\min \varepsilon^{(m)}(t, y) \geqslant c_0(t_*, x_*), \qquad G(\bar{t}, t_*, x_*, v_0(\cdot)) \tag{2.10}$$

From (2.10) it follows that $(t_*, x_*) \in E_{m+1}$.

2°. In this case we have $\bar{t} < t^\circ$. We assume that $c_0(t_*, x_*) = \varepsilon^{(m+1)}(t_*, x_*)$. Then for any $t^* \in [t_*, 1]$ and $v^*(\cdot) \in \{v(\cdot), [t_*, 1]\}$ it follows from the condition

$$\min_{G(t^*,\, t_*,\, x_*,\, v^*(\cdot))} \varepsilon^{(m)}(t^*, x) = \varepsilon^{(m+1)}(t_*, x_*) \qquad (2.11)$$

that $t^* \geqslant t^\circ$ and $v^*(t) = 2$ almost everywhere on $[t_*, t^*]$, $x^\circ(t^*) \in G(t^*, t_*, x_*, v^*(\cdot))$. But in this case $x^\circ(t^*) > -x_0(t^*)$, $c_0(t^*, x^\circ(t^*)) < c_0(t_*, x_*)$ and (see (2.11)) we have $\varepsilon^{(m+1)}(t_*, x_*) < c_0(t_*, x_*)$ which contradicts the assumption. Thus (see (2.9)) we have proven that

$$E_{m+1} = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_{m+1}(1 - t)\}$$

Taking into account (2.6) and (2.7) as well as Lemmas 2 and 3, we can show that the following theorem holds.

Theorem. Sets E_k, $k \in N_0$ and E_∞ are defined by the conditions

$$E_k = \{(t, x) : (t, x) \in \Lambda_1, |x| \geqslant a_k(1 - t)\}$$
$$E_\infty = S = \{(t, x) : (t, x) \in [0, 1) \times R^1 \, |x| \leqslant 1 - t\}$$

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STABILITY IN FIRST APPROXIMATION OF STOCHASTIC SYSTEMS WITH AFTEREFFECT

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The theorem on existence of the Liapunov functionals and the theorem on stability in first approximation for a stochastic differential equation with aftereffect are proved.

The suggestion of the replacement of Liapunov functions by functionals [1] in the investigation of the stability of ordinary differential equations with lag, has been widely utilized in dealing with determinate systems, as well as in the case of linear and nonlinear stochastic systems (see, e. g. [2 — 11]). Results concerning the stability in the first approximation were obtained for stochastic systems in [12 — 18] and others. Use of Liapunov functionals for the differential equations with aftereffect was first encountered in [1, 19, 20] where the inversion theorems were proved and conditions for the stability in first approximation